

No Information Can Be Conveyed By Certain Events: The Case of the Clever Widows of Fornicalia And the Stobon Oracle

Anand Venkataraman and Ray Kemp
Computer Science, IIEST
Massey University
New Zealand
<mailto:{A.Raman,R.Kemp}@massey.ac.nz>

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Abstract

In this short article, we look at an old logical puzzle, its solution and proof and discuss some interesting aspects concerning its representation in a logic programming language like Prolog. We also discuss an intriguing information theoretic aspect of the puzzle.

1 Introduction

In a certain village on the remote plains of Fornicalia there exist some men who are having affairs with the wives of other men. Now there is a gruesome custom in this village which requires a woman to kill her husband the morning after she discovers that he is having an affair with another woman. It also happens that every woman knows whether every other man is having an affair or not except

her own husband.¹ So life in this village goes on peacefully since no woman can know for sure that her own husband is cheating on her. Unfortunately, an Oracle from the pure and untainted shores of distant Stobon visits the village one day and proclaims that at least one man in this village is having an affair. What happens after this?

Readers are urged to think of a solution themselves before proceeding further. We first present the solution and its proof and finally go on to discuss the information theoretic aspects of this remarkable problem.² The first two sections of this short note will be of use as a reading for undergraduate Computer Science students learning its foundations. Its final two sections are independent of each other and may be read in any order. Section 3 is a nice Prolog representation of this proof that can be used to visualise and feel comfortable with it for those of us, including from time to time, the authors, who are still skeptical in spite of believing the underlying mathematics. Section 4 deals with an intriguing aspect of this problem to do with Information Theory. It is likely to be useful to graduate students or computing professionals being exposed to Information Theory for the first time. In any case, the article is bound to be of interest to anyone with a problem-solving bent of mind.

The answer to the above question is that n killings will take place on the

¹A further important fact that is often not emphasized in most renditions of this puzzle is that not only does each woman know the character of every other man in the village except her own husband, she also knows this to be true of all the women in the village. It is also assumed that the women in this village possess the intelligence required to make logical inferences and that they recognize this ability in each other.

²A reader has remarked that “In older kinder times, descriptions of problems of this sort were less grisly. A person would have a black spot painted on his forehead or not. No mirrors exist in the community, so each person would know if another had a black spot, but he could not *directly* know about himself.”

morning of the n th day after the Oracle’s seemingly innocuous but ultimately catastrophic proclamation, where n is the number of unfaithful men. If there are 100 unfaithful men, for example, then nothing will happen for 99 days, but on the 100th morning, all 100 of the unfaithful men will be killed!

2 Proof

Let a woman be said to “know” that her husband is unfaithful when she has reason to believe beyond doubt that he is having an affair. Since she can never have first-hand knowledge about the character of her own husband it follows that her knowledge in this matter is necessarily inferred fact. In the absence of this knowledge, she is forced to subscribe to what we call “the conservative hypothesis” that one’s own husband is faithful. If a woman rejects the conservative hypothesis, then it follows that she has determined her husband is unfaithful and will thus kill him the next morning.

Here is the inductive assertion we make on i , the number of unfaithful men in the village:

$S(i)$: If there are i unfaithful men in the village, then $i - 1$ mornings after the Oracle’s proclamation, each of the i wives of the i unfaithful men will have conclusive evidence to dismiss the conservative hypothesis about their husbands.

2.1 Basis

Consider the case when there is exactly one unfaithful man A in the village.

Until the Oracle's visit, A 's wife has no reason to believe that her own husband is unfaithful and will thus continue to subscribe to the conservative hypothesis.

Immediately after the Oracle's visit, however, A knows that at least one man in the village is unfaithful. Since it cannot be any of the others whom A knows to be faithful, A infers that it must be her own husband and so kills him on the morning of the first day following the Oracle's visit. It has taken A 's wife exactly 0 mornings to determine whether her husband was faithful or not. The basis is therefore true.

2.2 Inductive step

Assume that our assertion $S(i)$ is true for all numbers of unfaithful men less than n , i.e. for $0 < i < n$. Now suppose that the village has n unfaithful men.

We prove by complete induction on n that $S(i)$ still holds.

When $i = n$, there are two possible implications that could result from the conservative hypotheses of the Fornicalian women:

1. $n - 1$ men in the village are unfaithful
2. n men in the village are unfaithful

Statement 1 is the implication of the conservative hypothesis held by the wives of each of the n unfaithful men, since they know $n - 1$ other unfaithful men and have no reason to believe their own husbands are unfaithful. Statement

2 is the implication of the conservative hypothesis held by the rest of the women in the village.

By the inductive hypothesis, the believers of statement 1 will expect $n - 1$ killings to take place on the morning of the $n - 1$ st day after the Oracle's visit. However, this doesn't happen since each of the n women will expect to see the husbands of the other $n - 1$ women dead and not their own. Therefore, on the morning when this expected killing doesn't take place, they are forced to reject their conservative hypothesis and conclude there must be more than $n - 1$ unfaithful men in the village. Since each of the women know that exactly $n - 1$ other men are unfaithful, each concludes that the only possibility is that n men must be unfaithful and that the n th unfaithful man must be her own husband. She thus kills him on the morning of the n th day. It has taken the n wives of the n unfaithful men exactly $n - 1$ mornings to determine that their husbands were unfaithful. This completes the inductive step and so $S(i)$ is true for all n .

3 The proof in Prolog

Modelling the puzzle in Prolog is an interesting exercise. While it tackles the problem from a different angle to shed further light on it, it also tests how effective a logic programming language would be for representing and solving problems of this kind.

3.1 Prolog preliminaries

The main assertion, which we will call $s(N)$, is that when there are N unfaithful husbands then the N wives will discover this on day $N - 1$. This can be represented by a Prolog predicate $s(N)$ which will succeed if the assertion is true and will fail otherwise. N , of course, must be a positive number so a first attempt at specifying our goal might be this:

```
s(N), N > 0.
```

However, this will not do on two counts. First, since Standard Prolog (Deransart, Ed-Dbali, and Cervoni 1996) does not have either implicit or explicit typing of variables there is no guarantee that N will be a number during a call. Secondly, Prolog insists that N must have been instantiated to a constant value by the time the call $N > 0$ takes place. We don't wish to be this restrictive since we need to check that it is satisfied for all positive N , so will want to be able to make the call with N uninstantiated.

A way of solving both problems at once is to represent numbers in structured form (see, for example, Sterling and Shapiro (1994)). We will slightly amend their representation, and denote zero by 0, 1 by 0+1, 2 by 0+1+1, etc.

To recognise numbers in this form we set up a predicate, `natural_number(N)` which succeeds if N is a number in this form and fails otherwise. A recursive definition of this predicate is given below:

```
natural_number(N) :- var(N), !.
```

```
natural_number(0).
natural_number(N+1) :- natural_number(N).
```

Notice that we allow any uninstantiated variable to denote a number which is what is needed for the general case. The two operations that we need to be able to carry out on these numbers are checking for equality, and checking for one being less than another.

We cannot use the standard unifier operator, `=`, for checking equality since a call such as $N = N + 1$ will send most Prolog systems to sleep. This is because the variable on the left hand side also occurs in the expression on the right hand side. To get around this problem, we use the Standard Prolog predicate `unify_with_occurs_check` to ensure this kind of comparison fails and then create an operator `eq` for checking equality of numbers thus:

```
:op(700, xfx, eq).

0 eq 0.
A+1 eq B+1 :- unify_with_occurs_check(A,B),
              natural_number(A), natural_number(B).
```

An operator, `lt`, for checking whether one number is smaller than another can similarly be set up:

```
:op(700, xfx, lt).

0 lt K+1 :- natural_number(K).
A+1 lt B+1 :- \+ (A eq B), A lt B.
```

`\+` is the Standard Prolog symbol for ‘not provable’. Note that we need to check for A being equal to B . Otherwise the test will loop in this situation.

3.2 Proof representation

Moving on now to how the inductive assertion, $s(N)$, can be expressed in terms of the problem, we first create a predicate, `reject_conservative_hypothesis`, which indicates the number of days that it will take the betrayed women of Fornicalia to abandon their conservative hypothesis. This can be written as follows:

```
reject_conservative_hypothesis(I, N, Day)
```

where I is the inductive counter, N denotes the test case, and Day is the day on which the women can work out the truth. Now we express $s(N)$ in terms of `reject_conservative_hypothesis`:

```
s(N) :- reject_conservative_hypothesis(N, N, Day), N eq Day+1.
```

The above statement says that if there are N unfaithful husbands then the wives will discover this on day number $N - 1$. Note that, as it happens, we do not have to stipulate $N > 0$ since this is implicit in the term $N = Day + 1$.

By the inductive assumption, if the number of betrayed women, I , is less than N then they will discover this on day $I - 1$. This can be written:

```
reject_conservative_hypothesis(I, N, Day) :- I lt N, I eq Day+1.
```

Then, of course, the killings take place. We introduce a predicate called `kill` to indicate when this occurs:

```
kill(I, Day)
```

This happens, for all I , on the day after the women learn the truth. So we have:

```
kill(I, Day+1) :- reject_conservative_hypothesis(I, N, Day).
```

Now we need to determine when the betrayed women find out in the case when $I = N$. The N women in this case subscribe to the conservative hypothesis and therefore believe there are $N - 1$ unfaithful husbands. They will, therefore, expect these husbands to be found out on day $N - 2$, and to be killed on day $N - 1$. If this is so then the call:

```
?- kill(N, Day).
```

where $N = Day + 1$ will succeed. If it doesn't then they will know their own husbands are unfaithful. We can express this in the Prolog clause:

```
reject_conservative_hypothesis(N, N, Day) :- N eq Day+1,  
          \+ kill(N, Day).
```

Finally, we have to check that the basis is satisfied. The simplest way of doing this is to make the call:

```
s(0+1)
```

to check whether it succeeds. Not only this, but we must ensure it does not use the inductive rule. A trace of the program, run under SICStus Prolog

(Andersson et al. 1993) is shown in Figure 1. It verifies that, indeed, the proof does not use the inductive rule and so is satisfied.

```
| ?- s(0+1).
Call: s(0+1)
  Call: reject_conservative_hypothesis(0+1,0+1,Day)
    Call: 0+1 lt 0+1
      Call: 0+1 eq 0+1
      Exit: 0+1 eq 0+1
    Fail: 0+1 lt 0+1
    Call: 0+1 eq Day+1
      Exit: 0+1 eq 0+1
      Call: \+ kill(0+1,0)
        Call: kill(0+1,0)
        Fail: kill(0+1,0)
        Exit: \+ kill(0+1,0)
      Exit: reject_conservative_hypothesis(0+1,0+1,0)
    Call: 0+1 eq 0+1
      Exit: 0+1 eq 0+1
    Exit: s(0+1)
```

Figure 1: A program trace of the basis of the proof in SICStus Prolog

If we now make a call with any legal value of N , ($N = 0 + 1 + 1 + 1$, for example), then S will succeed. Indeed, if we make the general call:

```
?- s(N).
```

then the solution will be:

```
N = _ + 1
```

where “ $_$ ” is the anonymous variable denoting any value. Figure 2 shows a program trace for this general case. Thus we can infer that the assertion is true for all values of N .

```

| ?- s(N).
Call: s(N)
    Call: reject_conservative_hypothesis(N,N,Day)
        Call: N lt N
        Fail: N lt N
        Call: N eq Day+1
        Exit: Day+1 eq Day+1
        Call: \+ kill(Day+1,Day)
            Call: kill(Day+1,Day)
                Call: reject_conservative_hypothesis(Day+1+1,N,Day)
                    Call: Day+1+1 lt N
                    Fail: Day+1+1 lt N
                    Call: Day+1+1 eq Day+1
                    Fail: Day+1+1 eq Day+1
                        Fail: reject_conservative_hypothesis(Day+1+1,N,Day)
                        Fail: kill(Day+1,Day)
                        Exit: \+ kill(Day+1,Day)
                        Exit: reject_conservative_hypothesis(Day+1,Day+1,Day)
                    Call: Day+1 eq Day+1
                    Exit: Day+1 eq Day+1
    Exit: s(Day+1)

```

Figure 2: A program trace of the general case of proof in SICStus Prolog

Also, if we wish to find out which day the unfaithful husbands are discovered
then we can make the call

```
?- reject_conservative_hypothesis(N, N, Day).
```

which gives the general solution:

```
N = Day + 1 ,
Day = _
```

Clearly it is possible to represent the problem fairly directly in Prolog and also to use the language's deductive mechanism to verify the solution. However, the representation and manipulation of numbers is rather clumsy and the

program produced is not purely declarative. It may be that using a more sophisticated logic programming language such as Mercury (Somogyi et al. 1995) would produce a cleaner solution. A more significant limitation worth mentioning is that the program only confirms the solution — the insight enabling the problem to be solved in the first place has been provided by a human and is built into the program.

4 Information Content of the Proclamation

Surprisingly, the mathematical proof presented above requires the Oracle to very much be an essential part of this situation. If not for the Oracle, the basis fails. Yet, it seems on a first reading that the Oracle’s statement contains no useful information in a village that has more than one unfaithful man. Consider the village that has two unfaithful men, for example. Every woman in the village knows at least one unfaithful man. Thus the Oracle’s proclamation is at best, one could say, “stating the bleeding obvious.” A foundational result in Information Theory due to Shannon and Weaver (1949, p.82–83) shows that when the term *information* is sensibly defined, the information content of a symbol is equal to the negative logarithm of its probability. We also point the reader at another seminal work, Hamming (1980), where the former interesting result is more accessibly discussed at length, and at the URL: <http://cm.bell-labs.com/cm/ms/what/shannonday/> from where a copy of Shannon’s original paper can be downloaded. This latter document, however, doesn’t contain

Weaver's lucid discussion of the significance of Shannon's results.

Claiming that the information content of a symbol is related to the negative logarithm of its probability is tantamount to claiming that no useful information can be conveyed by a certain event. The more uncertain and therefore surprising an event is, the more information it contains. If we told you, for example, that the Sun rose in the East this morning, you will hopefully not benefit much from this fact by virtue of your already attaching a very high probability to this event. However, if we were to tell you instead that the Sun happened to rise in the West, that information, assuming it is true, will be of utmost interest and use to you. In general the information content in an event i is $-\log p_i$ where p_i is the probability of the event.

The term *information* is used somewhat differently in logic and in information theory. In the latter we are able to specify a single real number that uniquely characterises the information content of a source, but not so in the former. Yet, it turns out that Shannon's results are just as profoundly valid in logic or epistemology as they are elsewhere. For instance, each observer will attach a subjective probability to a given statement being true. And to this observer, the amount of information contained in an event that exposes the objective truth value of that statement is in fact equal to the negative logarithm of this subjective probability. So at the very least, if we know whether an event was certain or not, we can associate either zero information content with it or not. Naturally, we would expect an event void of information to have no effect upon its recipient and thus only expect effects to be caused by uncertain events,

however small this uncertainty is. The point that is relevant to our discussion here is that while we are unable to say precisely how much subjective information is contained in an uncertain event for each person, we are able to say with certainty that a sure event contains zero information for each person concerned.

Now suppose again that there are at least two unfaithful men in the village. Every wife in the village knows at least one unfaithful man and so the subjective probability she will assign to the event that there is at least one unfaithful man in the village is 1.0. It follows, therefore, that the information contained for her in the oracular statement that claims this certain event is $-\log 1 = 0$. So how can this event, seemingly void of information, cause such a catastrophic result? Let us rephrase the problem to make it even more explicit. **The Oracle has not told the women of Fornicalia anything new that they didn't already know. So why was its proclamation critical in determining the subsequent course of events in the village?** Pondering this question, more than any other, promises to be most instructive for someone just being introduced to the subtleties of Information Theory. Again, we invite the reader at this point to contemplate why this is so before proceeding further.

4.1 Why the Oracle is essential

It turns out that the Oracle's proclamation did indeed bear no useful information for any woman. And this is precisely the reason that no killing takes place on the first day following this event. But a fact about the proclamation itself bears useful information for just the wives of the two unfaithful men in the

village. In particular, it is the *meta-fact* that *the oracular proclamation had zero information for every woman in the village*. In other words, although every woman knew that the village had at least one unfaithful man, not every woman knew that “every woman knew that the village had at least one unfaithful man”. Consequently, these women will attach a non-unitary probability to this event and will thus find it useful. Let’s review the case when there are exactly two unfaithful men in the village. Call their wives A and B . Every woman in the village knows at least one unfaithful man. But not every woman knows that every woman knows at least one unfaithful man. To be precise, A and B know only one unfaithful man in the village and A does not know that B knows that there exists at least one unfaithful man and vice-versa. Thus the fact that B does know this comes as a surprise to A after the first morning. In general, one can make the following assertion for any number, $i > 0$, of unfaithful men in the village:

$T(i)$: If there are i unfaithful men in the village, then the wives of these i men don’t know (that every woman knows) $^{i-1}$ that there exists at least one unfaithful man in the village.

where the superscripted index denotes $i - 1$ repetitions of the parenthesized phrase. It is easy to prove $T(i)$ by induction along the lines of our previous proof in the first section. Thus those women who don’t know the above fact will attach a subjective probability $p < 1$ to it’s being true and consequently find it of informative value. So the Oracle is an integral part of the situation after all. Had it not visited the village, the catastrophe wouldn’t have been triggered off

and the village would have continued to exist as a harmonious society in which no woman can conclusively nail down her husband. However, that being not the case, thus ends the lamentable tale of the Stobon Oracle and the clever widows of Fornicalia.³ The men's lives wouldn't have been lost in vain if their story has inspired in new students a deep and lasting love for Information Theory.

References

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³Imagine the even more lamentable case when there is exactly one unfaithful man and his wife also happens to be the only free thinking secret rebel in the village who doesn't believe in its gruesome custom. Suppose she demonstrates her recalcitrance by not killing her husband on the morning of the first day after the Oracle's proclamation. What happens then?

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